A Geometric Approach to Solve Economic Models

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Abstract

An equilibrium in economics models can be often represented as a set of inter-temporal difference (or differential) equations. These equations involve states and controls variables and their solution are the so called policy functions. The general idea behind the method we propose here is to view these inter-temporal equations as geometric restrictions (manifolds), so that time shifted variables (in the present and/or in the past) are just others variables in a geometric space. Each manifold (correspondent to each equation) can be obtained without using any notion of time invariant policy function and it can be therefore computed separately from the others. For this reason this process is fully parallelizable. Moreover, it is usually fast because often each equation contains at least one variable that can be back out analytically. All manifolds are then intersected through sequential binary operations, this process is computationally efficient since it can be seen as a multi-steps univariate numerical optimization. The manifold that survives this process is a geometric restriction between the states and one control. Finally, the correspondent policy function can be unveiled imposing the fact that the variables in this surviving geometric restriction must be connected thorugh a time-invariant function.

In this paper we propose a novel method to solve dynamic stochastic models. The main insight is to view model optimality conditions as geometric restrictions rather than equations in time where future variables are implied by a time consistent policy function. Current numerical methods can be divided into two categories. On the one hand there are iteration based methods, such as VFI or EGM, which iteratively find the policy functions independently of initial conditions. This independence comes at a cost of having to evaluate or solve the same problem many times until

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convergence. On the other hand there are projection methods, which find the policy functions in one maximization step, but require to solve a highly dimensional multivariate optimization problem and most of the time works only given a reasonable guess. The method we introduce in this paper suggests to view the model optimality conditions as geometric restrictions in space rather than difference equations connected through Markov policy functions. In our method, each optimality condition is a restriction in the space of both endogenous and exogenous variables, while the joint information contained in all the optimality conditions is viewed as an intersection of the individual geometric restrictions. The method has multiple advantages. First, it does not depend on the initial guess. Secondly, it involves little use of numerical solver - geometric restrictions can be found without using the solver and the calculation of the intersection is a univariate optimization problem. Thirdly, the algorithm is easily parallelizable. A multivariate solver still needs to be used once - in finding the survival Markov policy.

Simple deterministic example

To gain intuition in a concrete setup we first solve the Neoclassical Growth Model in a deterministic setting. The equilibrium of the model can be summarized by the following two equations:

$$u'(c) = \beta\left[u'(c')\left(zK'^{\alpha-1} + 1 - \delta\right)\right]$$
$$c + K' - (1-\delta)K = zK^{\alpha}$$

In the first step we substitute out c from the Euler Equation (using the budget constraint) and obtain a geometric restriction between K, K' and K''. In the second step we take this restriction and explicitly impose that $K \to K'$ and $K' \to K''$ need to be connected by the same time invariant policy function. Parameter values an functional forms are reported in Appendix.

1. Create a grid for K and K', hence obtain K'' using the following equation:

   $$K'' = u'^{-1}\left[\frac{u'(zK^{\alpha} + (1-\delta)K - K')}{\beta(1-\delta+zK'^{\alpha-1})} - (F(K') + (1-\delta)K')\right]$$

   This gives an implicit restriction : $K'' = F^e(K', K)$

2. Impose the fact that policy is Markov: $K' = f(K)$ and find $f(.)$ from

   $$f(f(K)) = F^e(f(K), K)$$

In this exmple, we assume that:

$$K' = f(K) = \phi_1 + \phi_2 K + \phi_3 K^2$$

In general, note that $f(.)$ does not need to be parametric. Step (2) can as well be implemented using non parametric methods such as artificial neural networks.

Figure 1 represents the visual implementation of the algorithm.

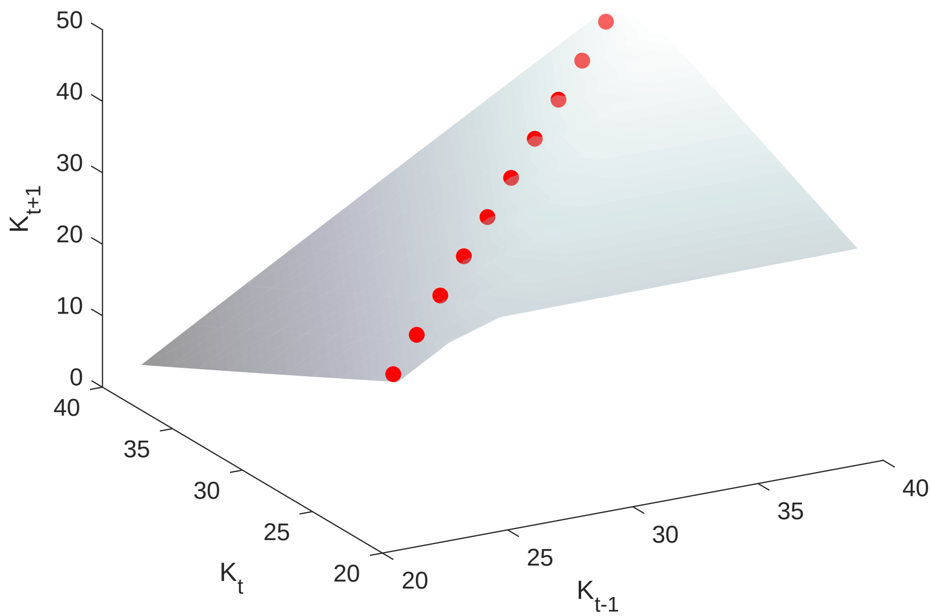

Figure 1: The combination of the Euler equation and the budget constraint can be represented by the manifold in the picture. The red dots are the points where the functions that connect K_{t-1} to K_t and K_t to K_{t+1} are the same.

Stochastic example

In this section we illustrate the algorithm in a stochastic version of the same model. Recall the optimality conditions:

$$u'(c) = \beta \mathbb{E}\left[u'(c')\left(z' F_K(K') + 1 - \delta\right)\right]$$

$$c' + K' - (1-\delta)K_t = zF(K)$$

In this example we obtain the geometric restrictions corresponding to the Euler equation and the budget constraint separately and then illustrate how to find their intersection. In the last step we use the manifold that survives the intersection to find the policy function by imposing a Markov restriction. Parameter values and functional forms are reported in Appendix.

1. Get the geometric restriction from the Euler equation. In a stochastic setting K'' is a function of future z. We impose this restriction by evaluating the Euler equation on separate grids for K'' when z is low ($z = z_1$) and z is high ($z = z_2$), denoted by K''^1 and K''^2 [1]. In the first step we set grids for z, K', K''^2 and K''^1 and find a geometric restriction:

$$C^E(K', K''^1, K''^2, z) = \beta \sum_{i=1,2} \pi(z_i|z) u'(F(K', z') + (1-\delta)K' - K''^i)(F_K(K', z') + 1 - \delta)$$

2. Get the geometric restriction from the budget constraint - find C in terms of K, K', z

$$C^B(K, K', z) = zF(K) + (1-\delta)K - K'$$

3. Intersect C^B and C^E

$$K'(K, K''^1, K''^2, z) \leftarrow C^E(K', K''^1, K''^2, z) \cap C^B(K, K', z)$$

More formally, the intersection is defined as[2]:

$$K' = F^k(K, K''^1, K''^2, z) = \arg\min_{K'} (C^E(K', K''^1, K''^2, z) - C^B(K, K', z))^2 \tag{1}$$

Figure 2 illustrates this step on the dimension of K' and z, keeping K, K''^1, K''^2 fixed. Two plains show the geometric restrictions on C and the red points indicate the points that solve equation 1.

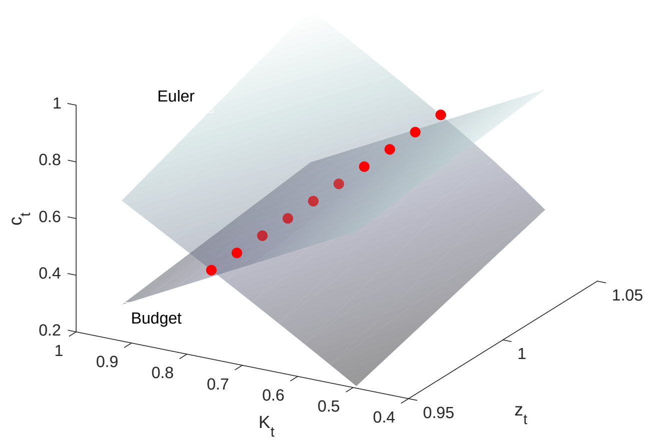

Figure 2: Intersection of manifolds coming from budget constraint and Euler equation

[1] z is assumed to take two values

[2] F^k only denotes a mapping between K, K''^1, K''^2, z and K

4. Compute Markov policy $K' = f(K, z)$:

   Assume that $f(K, z)$ has a parametric form $f(K, z, \phi)$ and solve the following functional minimization problem [3]:

$$\phi = \arg\min_{\phi} \Big(f(K, z, \phi) - F^k(K, f(f(K, z, \phi), z_1, \phi), f(f(K, z, \phi), z_2, \phi), z) \Big)^2$$

   Figures 3 and 4 illustrate this step. The slices of the multi-dimensional manifold in figure 3 present the restriction $K' = F^k(K, K''^1, K''^2, z)$ when $z = z_1$ and $K''^2 = \{K_1, K_3, K_5, K_7, K_{10}\}$ [4]. Red dots present the implied policy for K' and K''^2 starting from every point in the K grid. One can notice that the policy function lies in the space confined by the $K' = F^k(K, K''^1, K''^2, z)$ manifold. Figure 4 shows the analogous information fixing z at $z = z_2$, and $K''^1 = \{K_1, K_3, K_5, K_7, K_{10}\}$

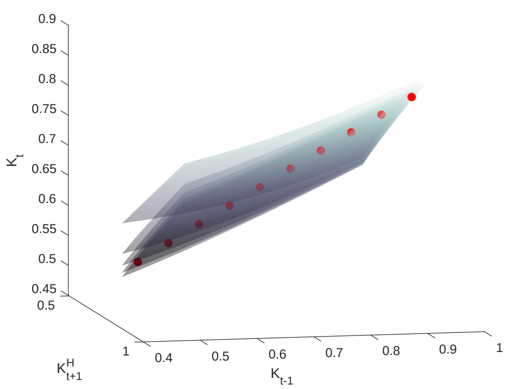

Figure 3: $z = z_1$, K''^2 - fixed

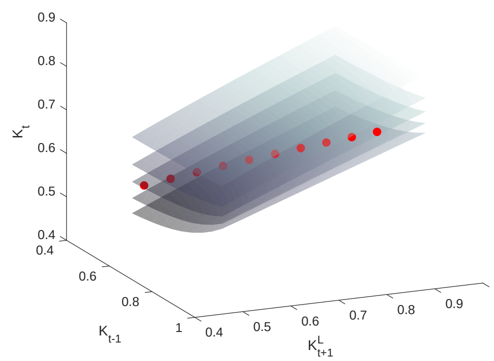

Figure 4: $z = z_2$, K''^1 - fixed

Generic Algorithm

In this section we present a generic version of this algorithm for a generic model characterized by a number j of optimality conditions and having $X = \{x1, x2, x3, x4...x_i\}$ variables (these variables include all controls, endogenous and exogenous states variables and all their lags). Define the manifold associated with the optimality condition j as $M^j(X)$. The algorithm is as follows:

1. For every optimality condition j find a manifold $M^j(X)$ implicitly defining the geometric restriction on a variable x_i in terms of X_{-i}, such that $M^j(X) = F(X_{-i}) - x_i = 0$. Note that this step can be fully parallelized across j.

[3] It is not necessary to make parametric assumption here. Yet in this simple model a non-parametric function would be unnecessary

[4] $K_1, K_3, K_5, K_7, K_{10}$ are points on the K''^2 grid

2. Iteratively find the pairwise intersections of the j manifolds. Denote $\tilde{M}^1(X)$ the manifold after intersecting the two $M^j(X)$ manifolds:

$$\tilde{M}^1(X) = M^{j^1}(X) \cap M^{j^2}(X)$$

Keep calculating the intersections until all information from the optimality conditions is exhausted:

$$\tilde{M}^i(X) = M^{j^{i+1}}(X) \cap \tilde{M}^{i-1}(X)$$

3. Solve for Markov policy

Define $\tilde{M}^i(X, f) \equiv F(X_{-i}, f) - f(x_i) = 0$ and find the policy function f

$$f = \arg\min_f (F(X_{-i}, f) - f(x_i))^2$$

Robustness

The method depends on finding geometric restrictions on independent grids. In general, problems arise when some combinations of grid points result in unfeasible values of endogenous variables. As a last exercise we solve a model when $\delta = 0.1$, in this case the grids are built such that the geometric restrictions imply negative consumption in a large number of grid points. For illustration purposes we assign K' to be 0 in proximity of the unfeasible points (otherwise complex numbers or NaN value will arise in these areas). As long as the intersection process and the Markov seeking procedure operate in a geometric area not intersected by this unfeasible area the procedure still works. Figures 5 and 6 illustrate this idea.

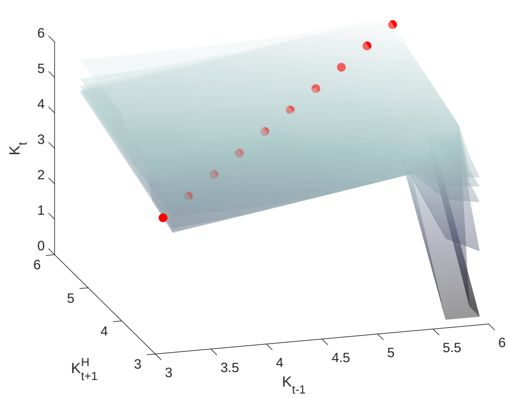

Figure 5: Policy with unfeasible grid points, $z = z_1$

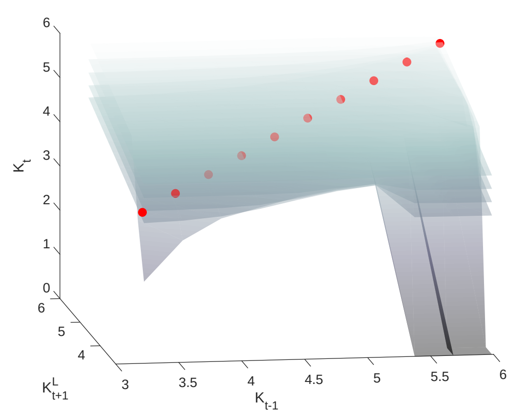

Figure 6: Policy with unfeasible grid points, $z = z_2$

Appendix

γ	β	α	δ	z_1, z_2	P_z
2	0.98	1/3	0.5	0.95, 1.05	$\begin{pmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{pmatrix}$

$F(K) = K^\alpha$, $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$