Intro to Numerical Methods

Class I: The Neoclassical Investment Model and VFI

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The Neoclassical Investment Model

Core model structure

Firms invest to maximize the present value of profits, equating the marginal benefit and cost of capital. Households own firms, supply labor, and consume dividends.

Key mechanism

Investment raises future productive capacity at the cost of foregone consumption.

Applications:

- Studying business investment cycles.
- Studying effects of monetary and fiscal policy.
- Understanding firm dynamics and growth.
- DSGE NK models, heterogeneous agents and firms.

The Firm's Problem

- **Firm owns physical capital K_t and chooses investment I_t.**
- Production function:
$$Y_t = F(K_t, N_t)$$
- Capital accumulation:
$$K_{t+1} = (1-\delta)K_t + I_t$$
- Profits:
$$D_t = F(K_t, N_t) - w_t N_t - I_t$$
- **Problem:**
$$\max_{\{I_t, N_t\}} \sum_{t=0}^{\infty} \beta^t D_t$$

First-Order Conditions

- **Labor demand:**

$$w_t = F_N(K_t, N_t)$$

  - Wages equal the marginal product of labor.

- **Capital Euler equation:**

$$1 = \beta \left[F_K(K_{t+1}, N_{t+1}) + (1 - \delta) \right]$$

  - The marginal cost of investing today equals the discounted marginal return to capital tomorrow.

Household Problem

Preferences (risk-neutral)

$$\max_{\{C_t, S_{t+1}, B_{t+1}\}_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t \, C_t$$

Budget constraint

$$C_t + p_t S_{t+1} + q_t B_{t+1} \;=\; w_t \bar{N} \;+\; (p_t + D_t) S_t \;+\; B_t$$

p_t: price of one share; D_t: dividend per share. S_t: shares; B_t: risk-free bond holdings. $\bar{N}$: inelastic labor supply (normalize to 1 if desired).

FOCs / Pricing:

$$p_t \;=\; \underbrace{\beta\,[D_{t+1} + p_{t+1}]}_{\text{NPV of dividends}}, \qquad \underbrace{q_t}_{\text{Risk-free bond price}} \;=\; \beta$$

Closing the Model

Market clearing conditions:

- Goods market clears: $Y_t = F(K_t, N_t)$.
- Labor market clears: $N_t = \bar{N}$.
- Capital market clears: $S_t = 1$.
- Bonds are in zero net supply: $B_t = 0$.

Resource constraint

Combine the market clearing conditions with the household budget constraint and the firm dividend to get: $C_t + I_t = Y_t = F(K_t, N_t)$.

From Competitive Equilibrium to Planner

Under perfect competition, constant returns, and no distortions, the competitive equilibrium allocation solves the planner's problem:

$$\max_{\{C_t, K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t C_t$$

subject to the single feasibility constraint (resource constraint)

$$C_t + K_{t+1} = ZK_t^{\alpha} + (1-\delta)K_t, \qquad K_0 \text{ given.}$$

This is equivalent to imposing market clearing on the household problem and substituting firm FOCs into prices.

From Sequential to Recursive Formulation

Sequential Planner Problem:

$$\max_{\{C_t, K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t C_t \quad \text{s.t.} \quad C_t + K_{t+1} - (1-\delta)K_t = ZK_t^{\alpha}$$

Step to Recursion:

- Take current capital $K \equiv K_t$ as the *state*.
- Choose next period's capital $K' \equiv K_{t+1}$ as the *control*.
- Use feasibility to solve for consumption:

$$C(K, K') = ZK^{\alpha} - \big(K' - (1-\delta)K\big).$$

- Rewrite the lifetime problem recursively:

$$V(K) = \max_{K' \geq 0} \Big\{ C(K, K') + \beta V(K') \Big\}.$$

Homework 1: Solve the following problem numerically...

Planner problem

$$V(K) = \max_{K' \geq 0} \Big\{ C(K, K') + \beta V(K') \Big\},$$

...subject to...

$$C(K, K') = ZK^{\alpha} - \big(K' - (1 - \delta)K\big).$$

Discount factor: $\beta = 0.96$. Capital share in production: $\alpha = \frac{1}{3}$.
Depreciation rate: $\delta = 0.10$. Productivity level: $Z = 1$.

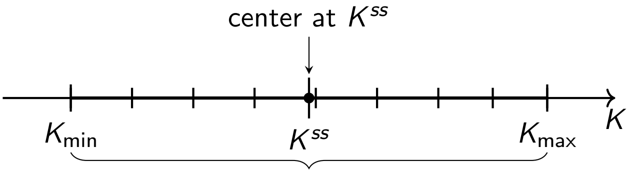
First Step: Grid Centered at the Steady State

Compute the steady state:

$$1 \;=\; \beta\big(F_K(K^{ss}) + 1 - \delta\big) \quad \Rightarrow K^{ss} \;=\; \left(\frac{\frac{1}{\beta} - (1-\delta)}{\alpha Z}\right)^{\frac{1}{\alpha - 1}}.$$

Center the grid around K^{ss}:

$$K_{\min} = (1-\eta)\,K^{ss}, \qquad K_{\max} = (1+\eta)\,K^{ss}, \qquad K_i \in \{K_{\min}, \dots, K_{\max}\}.$$



For this assignment I used $\eta = 0.99$ (very wide range)

VFI I: Maximization via Grid Search

Algorithm:

1. Initialize $V^{(0)}(K_i) = 0$ for all i.
2. For each K_i, evaluate

$$\text{RHS}(K_i, K_j) = C(K_i, K_j) + \beta V^{(n)}(K_j), \ \forall K_j \in \{K_{\min}, \ldots, K_{\max}\},$$

   and set $V^{(n+1)}(K_i) = \max_j \text{RHS}(K_i, K_j)$ with argmax policy $K'(K_i)$.
3. Stop when $\|V^{(n+1)} - V^{(n)}\| < \varepsilon$.

- *Pros:* simple, global maximizer.
- *Cons:* discrete K' (blocky policy) $\rightarrow$ could be imprecise.

VFI II: Interpolation + Golden Section Search

Idea

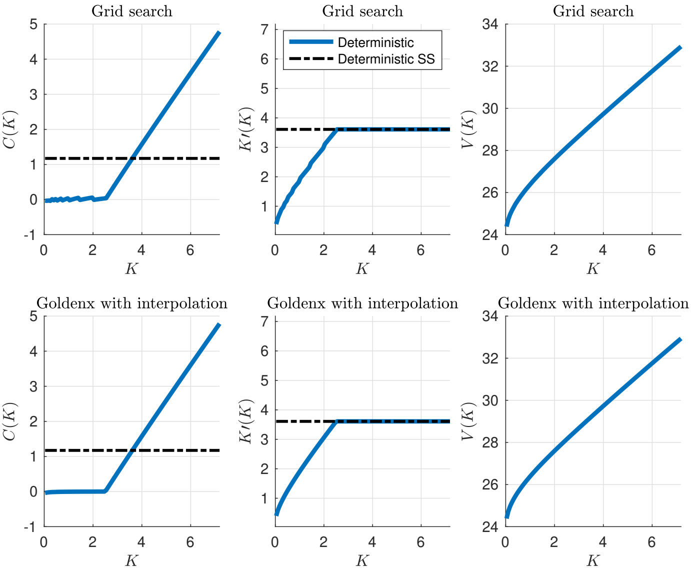
Keep the state grid for K, but treat the choice K' as *continuous* in $[K_{\min}, K_{\max}]$. Interpolate $V(K')$ between nodes and use a 1D optimizer.

Algorithm:

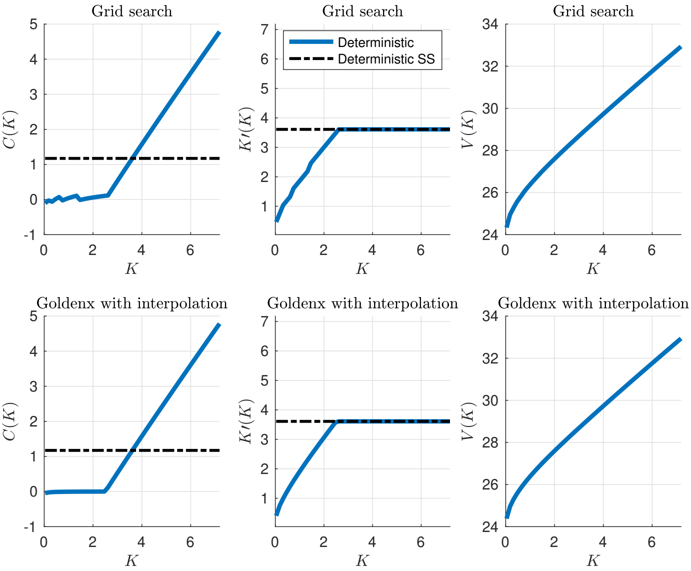
1. Fix K_i, define $g(K') = C(K_i, K') + \beta\, \widetilde{V}^{(n)}(K')$, where $\widetilde{V}$ is obtained via `interp1` (in Matlab).
2. Maximize $g(K')$ over $[K_{\min}, K_{\max}]$ using golden section search (`goldenx`) (in Matlab).
3. Update $V^{(n+1)}(K_i)$ and policy $K'(K_i)$. Iterate to convergence.

- *Pros:* smoother policies, higher accuracy, fewer evaluations.
- *Cons:* risk of local maxima.

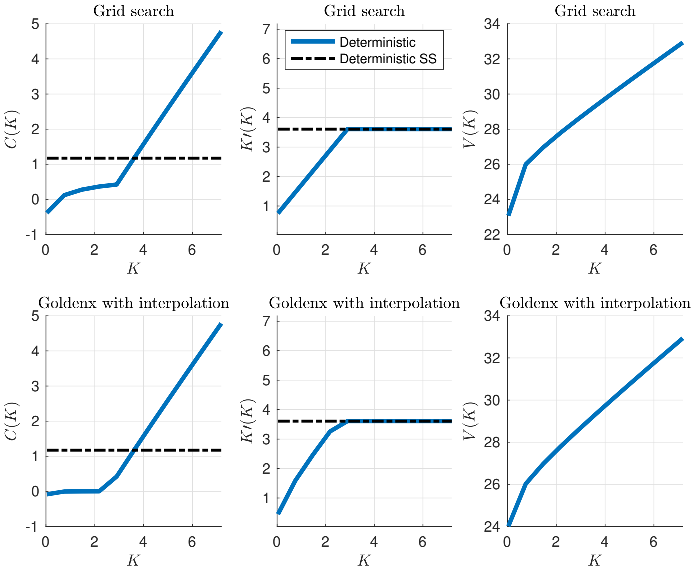
Preview of Results with 101 Nodes in the Capital Grid



Preview of Results with 51 Nodes in the Capital Grid



Preview of Results with 11 Nodes in the Capital Grid



Economic and Computational Takeaways

Result

- In a frictionless, risk-neutral world, firms face no reason to smooth investment and the economy leaps directly to its steady state.

But we are in general equilibrium...

- Consumption is nonnegative and investment is resource-limited.

Computational message

- Goldenx+interpolation performs well even with few nodes.

GOOD LUCK!

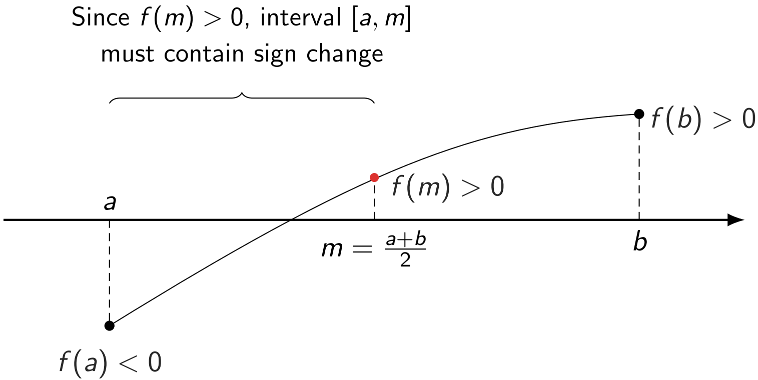
Golden Section Search

What is it?

A method to find the maximum (or minimum) of a function in a **1D interval** without derivatives. It works by repeatedly narrowing down the interval where the optimum lies.

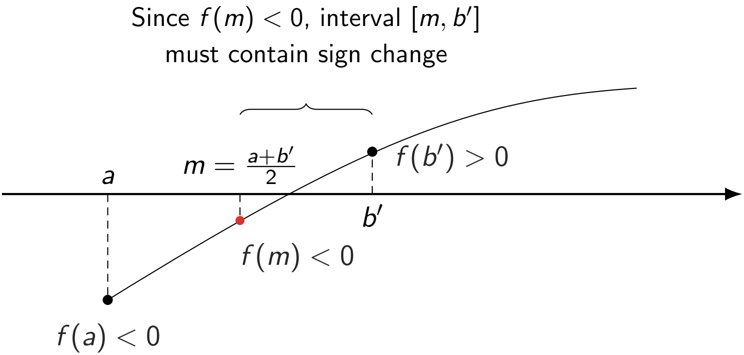
- For intuition, let's look at a simpler root-finding algorithm: the **bisection method**.

Dichotomic (Bisection) method — Step 1



Dichotomic (Bisection) method — Step 2

- **Now, reset endpoints: let $m = b'$ and repeat with new m.**



Golden Section Search

Key idea

Split the interval using the "golden ratio"

$$\varphi = \frac{\sqrt{5}-1}{2} \approx 0.618,$$

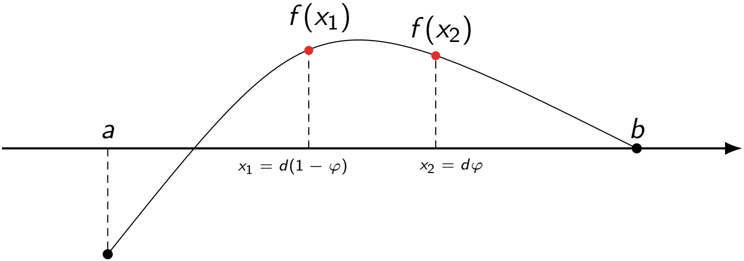
so that the function is only evaluated once per step.

Intuition

Think of "zooming in" on the peak: each iteration cuts the search interval smaller and smaller, always keeping the point with the higher value.

Golden Section Search — Step 1

- **Assumption:** f is unimodal on the search interval $[a, b]$.
- Define: $d \equiv b - a$.
- Pick x_1 and x_2 using the "golden ratio" φ.
- $f(x_1) > f(x_2) \implies$ maximum is to the left of x_2.



Golden Section Search — Step 2

- Redefine: $a = a', x_2 = b', x_1 = x_2', d' \equiv b' - a'$.
- Repeat process, setting new $x_1 = d(1 - \varphi)$.
- $f(x_1) < f(x_2) \implies$ maximum is to the right of x_1.