Intro to Numerical Methods

Class II: Real Business Cycle, Risk Aversion, Elastic Labor Supply

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Roadmap

1. Stochastic neoclassical investment model with elastic labor supply and CRRA preferences
2. Competitive equilibrium and planner equivalence
3. Recursive formulation with Markov shocks
4. Numerical solution: VFI with expectations (and labor)
5. Homework II

Environment

Technology

$$Y_t = F(K_t, N_t, Z_t) \; = \; Z_t \, K_t^{\alpha} N_t^{1-\alpha}, \qquad K_{t+1} = (1-\delta)K_t + I_t.$$

TFP shock drives business cycle → RBC model

Two states $Z_t \in \{Z_L, Z_H\}$ with transition matrix

$$\Pi = \begin{bmatrix} \pi_{LL} & \pi_{LH} \\ \pi_{HL} & \pi_{HH} \end{bmatrix}, \quad \Pr(Z_{t+1} = j \mid Z_t = i) = \pi_{ij}.$$

Preferences (CRRA over C, disutility of work)

$$u(C_t, N_t) = \frac{C_t^{1-\sigma} - 1}{1-\sigma} \; - \; \psi \, \frac{N_t^{1+\varphi}}{1+\varphi}, \qquad \sigma > 0, \; \psi > 0, \; \varphi \geq 0.$$

The Firm's Problem

- **Firm owns physical capital K_t and chooses investment I_t.**
- Production function:

$$Y_t = Z_t \, F(K_t, N_t)$$

- Capital accumulation:

$$K_{t+1} = (1 - \delta)K_t + I_t$$

- Profits:

$$D_t = Z_t \, F(K_t, N_t) - w_t N_t - I_t$$

- **Problem:**

$$\max_{\{I_t, N_t\}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} M_{0,t} \, D_t \right]$$

First-Order Conditions

- **Labor demand:**

$$w_t = Z_t F_N(K_t, N_t)$$

  - Wages equal the marginal product of labor.

- **Capital Euler equation:**

$$1 = \mathbb{E}_t\Big[M_{t,t+1}\big(Z_{t+1} F_K(K_{t+1}, N_{t+1}) + (1-\delta)\big)\Big]$$

  - The marginal cost of investing today equals the discounted marginal return to capital tomorrow.

Household Problem

Preferences

$$\max_{\{C_t,\, N_t,\, S_{t+1},\, B_{t+1}\}_{t\geq 0}} \; \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[u(C_t) - v(N_t)\right]$$

Budget constraint

$$C_t + p_t S_{t+1} + q_t B_{t+1} \;=\; w_t N_t \;+\; (p_t + D_t) S_t \;+\; B_t$$

p_t: price of one share; D_t: dividend per share. S_t: shares; B_t: risk-free bond holdings. N_t: elastic labor supply. Now w_t and D_t contain Z_t.

FOCs / Pricing:

$$p_t \;=\; \underbrace{\mathbb{E}_t[M_{t,t+1}\,(p_{t+1} + D_{t+1})]}_{\text{Expected NPV of dividends}}, \qquad \underbrace{q_t}_{\text{Risk-free bond price}} \;=\; \mathbb{E}_t[M_{t,t+1}]\,.$$

Household Problem: Elastic Labor Supply

With an elastic labor supply you have one more equation: the first-order condition with respect to N_t.

Intra-temporal Euler equation

Households work until the pain of supplying more labor ($v_n(N_t)$) is exactly balanced by the gain in consumption it provides ($u_c(C_t)w_t$).

$$u_c(C_t)w_t = v_n(N_t).$$

Note

If you prefer to avoid the complication of elastic labor supply, simply set $N_t = 1$ and drop the disutility of labor from preferences. This way you can focus solely on introducing the TFP shock and risk aversion.

Closing the Model

Market clearing conditions:

- Goods: $Y_t = Z_t F(K_t, N_t)$.
- Labor: N_t chosen by household (elastic), clears the labor market.
- Capital market: $S_t = 1$.
- Bonds: $B_t = 0$.

Resource constraint (as before)

Combine firm and household: $C_t + I_t = Y_t = Z_t F(K_t, N_t)$.

From Competitive Equilibrium to Planner

Under perfect competition, CRS, no wedges, the CE allocation solves:

$$\max_{\{C_t, N_t, K_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \psi \frac{N_t^{1+\varphi}}{1+\varphi} \right]$$

subject to (all t):

$$C_t + K_{t+1} = Z_t \, K_t^{\alpha} N_t^{1-\alpha} + (1-\delta) K_t, \quad K_0 \text{ given.}$$

Same logic as Class I: market clearing and firm FOCs replicate planner optimality; new parts are in red and preferences.

From Sequential to Recursive Formulation

Identify states and controls

State: Endogenous: $K \equiv K_t$. Exogenous: shock $Z \equiv Z_t \in \{Z_L, Z_H\}$.
Controls: $K' \equiv K_{t+1}$, $N \equiv N_t$.

Problem to solve

$$V(K, Z) = \max_{K' \geq 0,\, N \in [0,1]} \left\{ \frac{C^{1-\sigma} - 1}{1 - \sigma} - \psi \frac{N^{1+\varphi}}{1+\varphi} + \beta \sum_{Z'} \pi_{Z,Z'} V(K', Z') \right\},$$

s. to: $C(K, N, K'; Z) = Z\, K^{\alpha} N^{1-\alpha} - \big(K' - (1-\delta)K\big), \quad C \geq 0.$

Role of intra-temporal Euler

Directly optimize over N (using `goldenx`) OR solve: $u_c(C)w = v_n(N)$.

Homework II

Solve numerically the problem in the previous slide with the following parameters: $\beta = 0.96,\ \alpha = 1/3,\ \delta = 0.10,\ \sigma = 2,\ \varphi = 2$, choose ψ s.t. $N^{ss} \approx 1/3$ at $\bar{Z} = \mathbb{E}[Z].\ Z_L = 0.9,\ Z_H = 1.1,\ \Pi = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$.

Stationary distribution of Π

For a symmetric two-state Markov chain you have a 50-50 probability to be in either the L or H state in the long-run.

Long-run expectation

$$\mathbb{E}[Z] = 0.5 \cdot Z_L + 0.5 \cdot Z_H = 0.5 \cdot 0.95 + 0.5 \cdot 1.05 = 1.$$

First Step: Grid Centered at the (Average-Z) Steady State

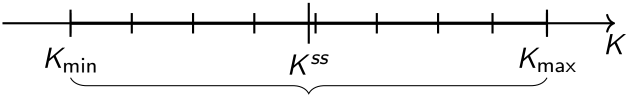
Deterministic reference: set $Z = \bar{Z} = \mathbb{E}[Z]$. Find (K^{ss}, N^{ss}) so that:

$$1 = M^{ss}\left(\alpha \bar{Z}\,(K^{ss})^{\alpha-1}(N^{ss})^{1-\alpha} + 1 - \delta\right), \quad M^{ss} = \beta,$$

$$\underbrace{\psi(N^{ss})^{\varphi}}_{v_n(N_t)} = \underbrace{\bar{Z}(1-\alpha)(K^{ss})^{\alpha}(N^{ss})^{-\alpha}}_{w_t}\underbrace{(C^{ss})^{-\sigma}}_{u_c(C^{ss})}.$$

Center the K-grid at K^{ss}:

$$K_{\min} = (1-\eta)\,K^{ss}, \qquad K_{\max} = (1+\eta)\,K^{ss}, \qquad K_i \in [K_{\min}, K_{\max}].$$



grid for example with $\eta = 0.2$

VFI I: Discrete K' Grid (global search)

Idea

Solve the deterministic model first and use it to initialize $V^{(0)}(K_i, Z_s)$.

Algorithm:

1. Initialize $V^{(0)}(K_i, Z_s) = V^{Deterministic}(K_i)$.
2. For each (K_i, Z_s), loop over $K'_j \in \{K_{\min}, \ldots, K_{\max}\}$.
   - For given K'_j, solve intratemporal FOC for $N \in [0, 1]$.
   - Compute C, utility $u(C, N)$. If $C \leq 0$, set value to $-\infty$.
   - Compute expectation $\sum_{Z'} \Pi_{Z,Z'} \cdot V^{(n)}(K'_j, Z')$.
3. Set $V^{(n+1)}(K_i, Z_s) = \max_j\{\cdot\}$ with argmax policy $K'(K_i, Z_s)$, and implied $N(K_i, Z_s)$.
4. Stop when $\|V^{(n+1)} - V^{(n)}\| < \varepsilon$.

VFI II: Interpolation + Golden Section (continuous K')

Idea

Keep the state grid for K and shocks Z, treat K' as *continuous* in $[K_{\min}, K_{\max}]$. Interpolate $V^{(n)}(K', Z')$ and maximize in 1D.

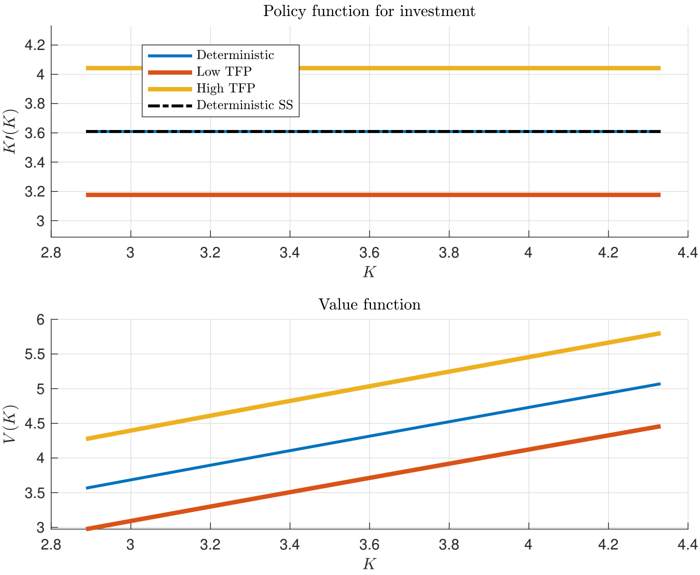
Algorithm:

1. Fix (K_i, Z_s). Define

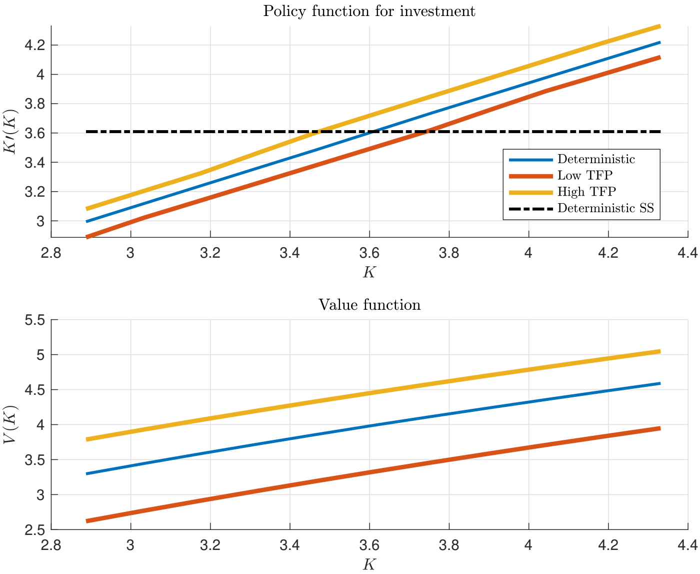
$$g(K') = u(C(K_i, N^\star(K', Z_s), K'; Z_s),\ N^\star(K', Z_s)) + \beta \sum_{Z'} \Pi_{Z,Z'}\, \widetilde{V}^{(n)}(K', Z'),$$

   where $N^\star$ solves the intratemporal FOC and $\widetilde{V}$ is interpolated.
2. Maximize $g(K')$ over $[K_{\min}, K_{\max}]$ (use `goldenx`).
3. Update $V^{(n+1)}(K_i, Z_s)$ and policies K', N. Iterate to convergence.

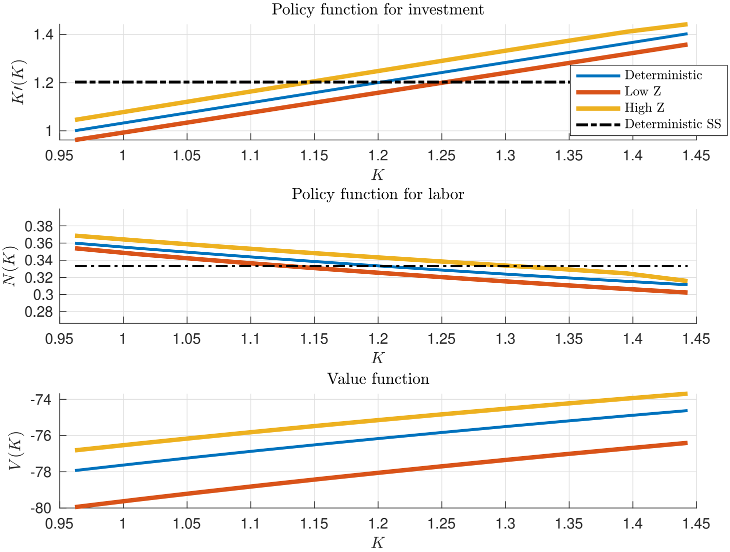
Results: risk neutrality ($\sigma = 0$) and inelastic labor ($N = 1$)



Results: risk aversion ($\sigma = 2$) and inelastic labor ($N = 1$)



Results: risk aversion ($\sigma = 2$) and elastic labor ($\varphi = 2$)



Economic Takeaway

Result

- Once shareholders are risk averse, they care about consumption smoothing.

Mechanism.

- Investing too aggressively would mean sacrificing dividends today, so identical firms accumulate capital gradually.

Overall message

- Dynamics arise because agents value a smoother consumption path rather than instant efficiency.

GOOD LUCK!