Intro to Numerical Methods

Class III: Nominal Rigidities, Technology Shocks, and Projection

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Roadmap

1. Stochastic neoclassical investment with elastic labor supply and CRRA preferences and nominal rigidities
2. Competitive equilibrium
3. Equilibrium Conditions
4. Numerical solution: Projection
5. Homework III

Environment I (same as class II)

Technology

$$Y_t = F(K_t, N_t, Z_t) \; = \; Z_t \, K_t^{\alpha} N_t^{1-\alpha}, \qquad K_{t+1} = (1-\delta)K_t + I_t.$$

TFP shock drives business cycle → RBC model

Two states $Z_t \in \{Z_L, Z_H\}$ with transition matrix

$$\Pi = \begin{bmatrix} \pi_{LL} & \pi_{LH} \\ \pi_{HL} & \pi_{HH} \end{bmatrix}, \quad \Pr(Z_{t+1} = j \mid Z_t = i) = \pi_{ij}.$$

Preferences (CRRA over C, disutility of work)

$$u(C_t, N_t) = \frac{C_t^{1-\sigma} - 1}{1-\sigma} \; - \; \psi \, \frac{N_t^{1+\varphi}}{1+\varphi}, \qquad \sigma > 0, \; \psi > 0, \; \varphi \geq 0.$$

Environment II (new)

New elements

Firms now choose the nominal prices at which they sell their goods. There is an aggregate price level P_t and gross inflation is defined as $\Pi_t \equiv \frac{P_t}{P_{t-1}}$.

Rotemberg adjustment costs

Firms incur a quadratic cost to adjust prices: $\mathcal{R}(\Pi_t) \equiv \frac{\varphi}{2}\left(\Pi_t - \bar{\Pi}\right)^2 Y_t$, where $\varphi > 0$ measures nominal rigidity and $\bar{\Pi}$ denotes the SS inflation rate.

Monetary policy (Taylor rule)

The central bank sets the gross nominal interest rate according to

$$1 + i_t = (1 + \bar{i}) \left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\phi_\pi} \left(\frac{Y_t}{\bar{Y}}\right)^{\phi_y}.$$

Final Good Producer (Dixit–Stiglitz aggregator)

- A competitive final good producer aggregates a continuum of differentiated varieties $Y_{i,t}$ into the final good Y_t:

$$Y_t = \left(\int_0^1 Y_{i,t}^{\frac{\theta-1}{\theta}} \, di \right)^{\frac{\theta}{\theta-1}}, \qquad \theta > 1.$$

- Given nominal prices $\{P_{i,t}\}_{i \in [0,1]}$, it chooses $\{Y_{i,t}\}$ to minimize cost

$$\min_{\{Y_{i,t}\}} \int_0^1 P_{i,t} \, Y_{i,t} \, di \quad \text{s.t.} \quad \left(\int_0^1 Y_{i,t}^{\frac{\theta-1}{\theta}} di \right)^{\frac{\theta}{\theta-1}} \geq Y_t.$$

- FOC $\Rightarrow$ demand for variety i and the aggregate price index:

$$Y_{i,t} = \left(\frac{P_{i,t}}{P_t} \right)^{-\theta} Y_t, \qquad P_t = \left(\int_0^1 P_{i,t}^{1-\theta} \, di \right)^{\frac{1}{1-\theta}}. \tag{1}$$

- Under perfect competition, the final good firm earns zero profits:

$$P_t Y_t = \int_0^1 P_{i,t} \, Y_{i,t} \, di.$$

The Firm i's Problem (with nominal rigidities)

- Firm i owns physical capital $K_{i,t}$ and chooses investment $I_{i,t}$, labor $N_{i,t}$, **and its nominal price** $P_{i,t}$.
- Production function and capital accumulation:

$$Y_{i,t} = Z_t\, F(K_{i,t}, N_{i,t}) \quad \text{and} \quad K_{i,t+1} = (1-\delta)K_{i,t} + I_{i,t}. \qquad (2)$$

- Profits (in nominal terms): $P_{i,t}\, Y_{i,t} \; - \; W_t\, N_{i,t} \; - \; P_t\, I_{i,t} \; - \; P_t\, \mathcal{R}_{i,t}$.
- Real profits (divide by P_t; let $w_t \equiv W_t/P_t$):

$$D_{i,t} \; = \; \left(\frac{P_{i,t}}{P_t}\right) Y_{i,t} \; - \; w_t N_{i,t} \; - \; I_{i,t} \; - \; \mathcal{R}_{i,t}.$$

In a symmetric equilibrium $P_{i,t} = P_t$ so $\frac{P_{i,t}}{P_t} = 1$, but not ex-ante.

- **Problem:** $\max_{\{I_{i,t},\, N_{i,t},\, P_{i,t}\}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} M_{0,t}\; D_{i,t}\right]$, s. to (1) and (2).

First-Order Conditions I

- **Labor demand (with final-good multiplier Γ_t):**

$$w_t \;=\; \Gamma_t\, Z_t\, F_N(K_t, N_t)$$

  - The real wage equals *marginal revenue product of labor*: Γ_t is the multiplier on the final-good demand constraint.

- **Capital Euler equation (with Γ_{t+1}):**

$$1 \;=\; \mathbb{E}_t\Big[\, M_{t,t+1}\Big(\Gamma_{t+1}\, Z_{t+1}\, F_K(K_{t+1}, N_{t+1}) \;+\; (1-\delta)\Big)\Big]$$

  - The marginal cost of investing today equals the discounted marginal return, with Γ_{t+1} scaling next period's marginal product.

First-Order Conditions II

- **Price-setting (Rotemberg NKPC, non-linear):**

$$Y_t \cdot \left(1 - \theta + \theta \Gamma_t \right) - \mathcal{R}_\pi(\Pi_t)\, \Pi_t + \mathbb{E}_t\Big[M_{t,t+1}\, \mathcal{R}_\pi(\Pi_{t+1})\, \Pi_{t+1} \Big] = 0,$$

where

$$\mathcal{R}_\pi(\Pi_t) \equiv \varphi\big(\Pi_t - \bar{\Pi}\big)\, Y_t, \qquad \theta > 1 \text{ is the elasticity of substitution.}$$

  - First term: desired markup vs. marginal cost (in your notation).
  - Middle/last terms: current and expected *Rotemberg* adjustment costs, discounted by $M_{t,t+1}$.

Household Problem

Preferences

$$\max_{\{C_t, N_t, S_{t+1}, B_{t+1}\}_{t\geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[u(C_t) - v(N_t)\right]$$

Budget constraint (in real terms)

$$C_t + p_t S_{t+1} + q_t B_{t+1} \;=\; w_t N_t \;+\; (p_t + D_t) S_t \;+\; \frac{B_t}{\Pi_t}$$

p_t: price of one share. D_t: dividend per share. S_t: shares; B_t: risk-free bond holdings. N_t: elastic labor supply. q_t: price of nominal bond.

FOCs / Pricing:

$$p_t \;=\; \mathbb{E}_t[M_{t,t+1}\,(p_{t+1} + D_{t+1})]\,, \quad (1 + i_t)^{-1} \equiv q_t \;=\; \mathbb{E}_t\left[M_{t,t+1}\Pi_t^{-1}\right].$$

Household Problem: Elastic Labor Supply

With an elastic labor supply you have one more equation: the first-order condition with respect to N_t.

Intra-temporal Euler equation

Households work until the pain of supplying more labor ($v_n(N_t)$) is exactly balanced by the gain in consumption it provides ($u_c(C_t)w_t$).

$$u_c(C_t)w_t = v_n(N_t).$$

Closing the Model

Market clearing conditions:

- Goods: $Y_t = Z_t F(K_t, N_t)$.
- Labor: N_t chosen by household (elastic), clears the labor market.
- Capital market: $S_t = 1$.
- Bonds: $B_t = 0$.

Resource constraint

Combine firm and household: $C_t + I_t + \mathcal{R}(\Pi_t) = Y_t = Z_t F(K_t, N_t)$.

Equilibrium

Given state variables $X_t \equiv \{K_t, Z_t\}$, the equilibrium is a set of policy functions $\{N(X_t), \Pi(X_t), K'(X_t), \Gamma(X_t)\}$ that solve:

1. $1 = \mathbb{E}_t\Big[M_{t,t+1}\Big(\Gamma(X_{t+1})\, Z_{t+1}\, F_K(K'(X_t), N(X_{t+1})) + (1-\delta)\Big)\Big],$
2. $u_c(C(X_t))w(X_t) = v_n(N(X_t)),$
3. $Y(X_t)\cdot\Big(1-\theta + \theta\Gamma(X_t)\Big) - \mathcal{R}_\pi(\Pi(X_t))\,\Pi(X_t) + \mathbb{E}_t\Big[M_{t,t+1}\, \mathcal{R}_\pi(\Pi(X_{t+1}))\,\Pi(X_{t+1})\Big]$
4. $q(X_t) = \mathbb{E}_t\left[M_{t,t+1}\, (\Pi(X_{t+1}))^{-1}\right],$

where:

1. $Y(X_t) = Z_t K_t^{\alpha} N(X_t)^{1-\alpha},$
2. $C(X_t) = Y(X_t) - (K'(X_t) - (1-\delta)K_t) - \mathcal{R}(\Pi(X_t)),$
3. $W(X_t) = \Gamma(X_t) Z_t K_t^{\alpha}(1-\alpha)N(X_t)^{-\alpha},$
4. $q(X_t) = \frac{\beta}{\bar{\Pi}}\left(\frac{\Pi(X_t)}{\bar{\Pi}}\right)^{-\phi_\pi}\left(\frac{Y(X_t)}{\bar{Y}}\right)^{-\phi_y}.$

Homework III

Solve numerically the equilibrium in the previous slide with the following parameters: $\beta = 0.96,\ \alpha = 1/3,\ \delta = 0.10,\ \sigma = 2,\ \varphi = 2$, choose ψ s.t. $N^{ss} \approx 1/3$ at $\bar{Z} = \mathbb{E}[Z]$. $Z_L = 0.99,\ Z_H = 1.01,\ \Pi = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$.

Nominal rigidities and elasticity parameters

Set $\varphi = 5$ and $\theta = 7$.

Taylor Rule parameters

Set $\phi_\pi = 1.5$ (Taylor principle) and $\phi_Y = 0.3$.

Numerical Solution: Projection Overview

- Unknown policy functions on states $X = (K, Z)$:

$$N(K, Z), \quad \Pi(K, Z), \quad K'(K, Z), \quad \Gamma(K, Z).$$

- Solve the steady state and build a grid for K as in the previous class. Use the steady state values to initialize your policy functions.
- Approximate each policy with linear polynomials in K (separately by Z state).
- Choose coefficients to make the model's equilibrium conditions hold at the nodes in the grid.

Steady-State System

1. Technology and resource constraint

$$\bar{Y} = \bar{Z}\,\bar{K}^{\alpha}\,\bar{N}^{1-\alpha}, \qquad \bar{Y} = \bar{C} + \bar{I}, \qquad \bar{I} = \delta\,\bar{K}.$$

2. Household optimality (intra-temporal condition)

$$u_c(\bar{C})\,\bar{w} = v_n(\bar{N}), \quad u_c(C) = C^{-\sigma}, \quad v_n(N) = \psi N^{\varphi} \Rightarrow \quad \bar{w} = \psi\,\bar{C}^{\sigma}\,\bar{N}^{\varphi}.$$

3. Firms (labor and capital conditions)

$$\bar{w} = \bar{\Gamma}\,\bar{Z}\,F_N(\bar{K}, \bar{N}) = \bar{\Gamma}\,\bar{Z}\,(1-\alpha)\,\bar{K}^{\alpha}\,\bar{N}^{-\alpha},$$

$$1 = \beta\Big[\bar{\Gamma}\,\bar{Z}\,F_K(\bar{K}, \bar{N}) + 1 - \delta\Big] = \beta\Big[\bar{\Gamma}\,\bar{Z}\,\alpha(\tfrac{\bar{Y}}{\bar{K}}) + 1 - \delta\Big].$$

4. Price-setting (Rotemberg NKPC in steady state)

Since $\Pi_t = \bar{\Pi}$, adjustment costs vanish:

$$0 = \bar{Y}\,(1 - \theta + \theta\,\bar{\Gamma}) \quad \Rightarrow \quad \boxed{\bar{\Gamma} = \tfrac{\theta-1}{\theta}}.$$

Policy Approximations (Quadratic in K by Z state)

Quadratic forms (when $Z = Z_L$)

$$N(K, Z_L) = a_N^L \, + \, b_N^L \, K \, + \, c_N^L \, K^2,$$
$$\Pi(K, Z_L) = a_\Pi^L \, + \, b_\Pi^L \, K \, + \, c_\Pi^L \, K^2,$$
$$K'(K, Z_L) = a_{K'}^L \, + \, b_{K'}^L \, K \, + \, c_{K'}^L \, K^2,$$
$$\Gamma(K, Z_L) = a_\Gamma^L \, + \, b_\Gamma^L \, K \, + \, c_\Gamma^L \, K^2.$$

Quadratic forms (when $Z = Z_H$)

$$N(K, Z_H) = a_N^H \, + \, b_N^H \, K \, + \, c_N^H \, K^2,$$
$$\Pi(K, Z_H) = a_\Pi^H \, + \, b_\Pi^H \, K \, + \, c_\Pi^H \, K^2,$$
$$K'(K, Z_H) = a_{K'}^H \, + \, b_{K'}^H \, K \, + \, c_{K'}^H \, K^2,$$
$$\Gamma(K, Z_H) = a_\Gamma^H \, + \, b_\Gamma^H \, K \, + \, c_\Gamma^H \, K^2.$$

Projection Step: Residuals and Updates

Initialize the 24 coefficients (from the SS). For each node (K_m, Z_L) and (K_m, Z_H):

1. **Evaluate future functions.**
   Use your current coefficients to approximate all policy functions that appear with X_{t+1}: $N'(X_{t+1}), \quad \Pi'(X_{t+1}), \quad K''(X_{t+1}), \quad \Gamma'(X_{t+1})$. Expectations over $Z' \in \{Z_L, Z_H\}$ use the transition matrix Π.

2. **Solve the four equilibrium conditions at each node.**
   For the given (K_m, Z_ℓ), solve jointly for the current values $\{\, N_m,\ \Pi_m,\ K'_m,\ \Gamma_m \,\}$ to zero the four residuals. *This is a nonlinear system of four equations per node.* In Matlab, you can solve it with `fsolve`.

3. **Update policy coefficients.**
   Once all $\{N_m, \Pi_m, K'_m, \Gamma_m\}$ are obtained on the grid:
   - Regress each function on K_m separately for $Z = Z_L$ and $Z = Z_H$:
     $$N(K_m, Z_\ell) = a_N^\ell + b_N^\ell K_m, \quad \Pi(K_m, Z_\ell) = a_\Pi^\ell + b_\Pi^\ell K_m, \text{ etc.}$$
   - Update the 24 coefficients $\{a, b, c\}$ and iterate until convergence. Stop when successive coefficients change less than a tolerance (e.g. $|\Delta a|, |\Delta b| < 10^{-5}$) or the residuals at all nodes are near zero.

Simulating the TFP Process (2-state Markov chain)

Inputs: states $\{Z_L, Z_H\}$, transition matrix $\Pi = \begin{bmatrix} \pi_{LL} & \pi_{LH} \\ \pi_{HL} & \pi_{HH} \end{bmatrix}$, horizon T, initial state Z_0, use a fixed seed for replicability, e.g. in Matlab `rng(100,"twister")`.

Inverse–CDF update rule (for $t = 0, 1, \ldots, T-1$)

Draw $u_t \sim \mathcal{U}(0,1)$ and set

$$Z_{t+1} = \begin{cases} Z_L, & \text{if } Z_t = Z_L \text{ and } u_t \leq \pi_{LL}, \\ Z_H, & \text{if } Z_t = Z_L \text{ and } u_t > \pi_{LL}, \\ Z_L, & \text{if } Z_t = Z_H \text{ and } u_t \leq \pi_{HL}, \\ Z_H, & \text{if } Z_t = Z_H \text{ and } u_t > \pi_{HL}. \end{cases}$$

- Start from Z_0 (e.g., just choose L or H).
- Save $\{Z_t\}_{t=0}^{T-1}$ to feed into the policy simulation.

Simulating the Model Using the Policy Functions

Given: converged quadratic policy approximations for each $Z \in \{Z_L, Z_H\}$,

$$N(K, Z_\ell) = a_N^\ell + b_N^\ell K + c_N^\ell K^2, \quad \Pi(K, Z_\ell) = a_\Pi^\ell + b_\Pi^\ell K + c_\Pi^\ell K^2,$$
$$K'(K, Z_\ell) = a_{K'}^\ell + b_{K'}^\ell K + c_{K'}^\ell K^2, \quad \Gamma(K, Z_\ell) = a_\Gamma^\ell + b_\Gamma^\ell K + c_\Gamma^\ell K^2.$$

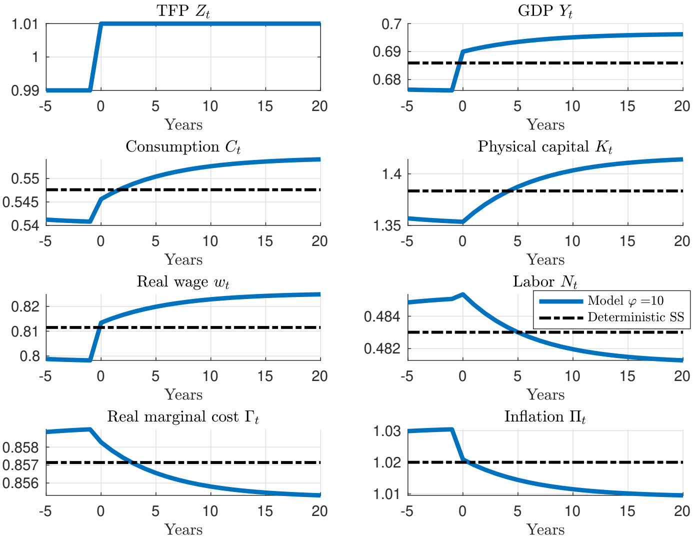
1. **Initialize:** $K_0 = \bar{K}$ (SS) and Z_0 from the Markov chain.
2. **For** $t = 0, 1, \dots, T-1$ with state $(K_t, Z_t = Z_\ell)$:

   Evaluate policies: $N_t = N(K_t, Z_\ell),\ \Pi_t = \Pi(K_t, Z_\ell),$
   $$\Gamma_t = \Gamma(K_t, Z_\ell),\ K_{t+1} = K'(K_t, Z_\ell).$$

   Aggregate quantities: $Y_t = Z_t K_t^\alpha N_t^{1-\alpha}, \qquad I_t = K_{t+1} - (1-\delta)K_t,$
   $$w_t = \Gamma_t\, Z_t\,(1-\alpha)\, K_t^\alpha N_t^{-\alpha},$$
   $$\mathcal{R}(\Pi_t) = \frac{\varphi}{2}\,(\Pi_t - \bar{\Pi})^2\, Y_t, \qquad C_t = Y_t - I_t - \mathcal{R}(\Pi_t).$$

   Policy block: $1 + i_t = (1+\bar{i})\left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\phi_\pi}\left(\frac{Y_t}{\bar{Y}}\right)^{\phi_y}, \qquad q_t = \dfrac{1}{1+i_t}.$
3. **Advance the shock:** draw Z_{t+1} via the Markov rule (previous slide).

Simulate a positive TFP shock



Economic Takeaway

A positive productivity shock ($Z_t \uparrow$)

- Firms are more efficient $\rightarrow$ can produce more with the same inputs.
- Output Y_t, capital K_t, consumption C_t, and real wages w_t all rise.
- Labor N_t falls: people can enjoy more income while working less.

Prices adjust slowly (Rotemberg nominal rigidities)

- Higher productivity lowers firms' marginal costs ($\Gamma_t \downarrow$).
- Firms face less pressure to raise prices $\rightarrow$ inflation Π_t **falls**.
- The economy experiences a smooth, disinflationary expansion.

Overall message

- **Supply-driven boom**: output and welfare rise while inflation declines.

GOOD LUCK!