Intro to Numerical Methods — Homework 2 Guide

Class II: Real Business Cycle, Risk Aversion, Elastic Labor Supply

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Learning Goals

In this second assignment you will:

1. Extend the deterministic model to include risk aversion and uncertainty in productivity.
2. Implement Value Function Iteration with expectations over future TFP states.
3. (Part B) Endogenize labor supply and compute the joint policy functions for K' and N.

1 Economic Environment

We consider a representative agent with preferences:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(C_t), \quad u(C) = \frac{C^{1-\gamma} - 1}{1-\gamma}, \quad \gamma > 0.$$

Production follows a Cobb–Douglas technology:

$$Y_t = Z_t K_t^{\alpha} N_t^{1-\alpha},$$

with capital law of motion

$$K_{t+1} = (1-\delta)K_t + I_t, \quad C_t = Y_t - I_t.$$

Labor is first exogenous ($N_t = 1$), and then endogenous in Part B.

2 Model Parameters

Use:

$$\beta = 0.96, \quad \alpha = \tfrac{1}{3}, \quad \delta = 0.10, \quad \gamma = 2, \quad Z \in \{0.9, 1.1\}.$$

The two TFP states follow a Markov chain with transition matrix

$$P_Z = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}.$$

```
% Preferences and technology
p.beta  = 0.96;
p.alpha = 1/3;
p.delta = 0.10;
p.gamma = 2;

% TFP process
Z_grd = [0.9, 1.1];
PZ = [0.9 0.1; 0.1 0.9];
```

Listing 1: Parameter block for both models

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3 Part A — Inelastic Labor Supply

In this version, labor is fixed ($N = 1$). The agent decides only over next period's capital K'.

Deterministic Baseline

a) Compute the steady-state capital K^*:

$$K^* = \left[\frac{1/\beta - 1 + \delta}{Z\alpha}\right]^{1/(\alpha-1)}.$$

b) Build a symmetric grid around K^* (use `NumNodes=11` and `Explore=0.2`).

c) Initialize $V(K)$ to zeros and iterate using a golden-section search over K'.

```
for each K in K_grid
% Define consumption as a function of Kprime
currC = @(Kp) F(K) - (Kp - (1 - p.delta)*K);
% Objective: u(C) + beta * V(Kp)
obj   = @(Kp) u_safe(currC(Kp), p) + p.beta * interp1(K_grid, V, Kp);
% Maximize using goldenx
[Kprime_opt(iK,1), Vnew(iK,1)] = goldenx(obj, Kmin, Kmax);
end
```

Iterate until $\|V^{t+1} - V^t\|_\infty < 10^{-5}$.

Stochastic Extension

Now include productivity shocks:

$$Z_t \in \{Z_L, Z_H\} = \{0.9, 1.1\}, \quad P(Z'|Z) = P_Z.$$

The Bellman equation becomes

$$V(K, Z) = \max_{K'} \{u(C(K, Z, K')) + \beta \sum_{Z'} P_Z(Z'|Z)\, V(K', Z')\}.$$

Implementation Steps

a) Initialize $V(K, Z_L)$ and $V(K, Z_H)$ using the deterministic $V(K)$.

b) For each (K, Z), compute expected continuation value:

$$E[V(K', Z')|Z] = \sum_{Z'} P_Z(Z'|Z)\, V(K', Z').$$

c) Maximize the Bellman objective with respect to K' using `goldenx`.

d) Store policy functions $K'(K, Z)$ and updated $V(K, Z)$.

```
    for each K in K_grid
    for each Z in Z_grid
    currC = @(Kp) F(K,Z) - (Kp - (1 - p.delta)*K);
    EV = @(Kp) PZ(Z,1)*interp1(K_grid,V(:,1),Kp) + ...
    PZ(Z,2)*interp1(K_grid,V(:,2),Kp);
    obj = @(Kp) u_safe(currC(Kp),p) + p.beta*EV(Kp);
    [Kprime_opt(K,Z), Vnew(K,Z)] = goldenx(obj, Kmin, Kmax);
    end
    end
```

Plots

Plot policy and value functions for low and high Z. Label steady state K^* for comparison.

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4 Part B — Endogenous Labor Supply

Now labor N becomes a choice variable. The agent chooses both K' and N:

$$V(K, Z) = \max_{K', N} \{\, u(C(K, N, Z, K')) - v(N) + \beta E[V(K', Z')] \,\}.$$

The disutility of labor is

$$v(N) = \chi \frac{N^{1+\frac{1}{\phi}}}{1 + \frac{1}{\phi}},$$

and period utility is $u(C, N) = u(C) - v(N)$.

Parameter Additions

$$\chi = 11.6, \quad \phi = 2.$$

Labor bounds: $N \in [10^{-6}, 0.999]$.

```
    p.phi = 2;     % Frisch elasticity
    p.chi = 11.6; % Disutility weight

    N_min = 1e-6;
    N_max = 0.999;
```

Steady-State Centering

Compute (K^*, N^*, C^*) analytically via:

$$\begin{aligned} r &= 1/\beta - (1-\delta), \\ \kappa &= (r/(\alpha Z))^{1/(\alpha-1)}, \quad w = (1-\alpha) Z \kappa^{\alpha}, \\ N^* &= \Big(\frac{w A^{-\gamma}}{\chi}\Big)^{1/(\gamma+1/\phi)}, \quad K^* = \kappa N^*. \end{aligned}$$

You may implement this in a helper function `steady_state_endolabor_closedform.m`.

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Deterministic Model (Endogenous Labor)

Algorithm outline:

a) For each K on the grid:

- For each K' candidate, solve the *inner problem* for N via a golden-section search:

$$\max_N u(C(K, N, 1, K')) - v(N).$$

- Plug the resulting utility into the outer Bellman objective:

$$\text{obj}(K') = u(C, K', N^*(K')) + \beta V(K').$$

- Maximize over K' using `goldenx`.

b) Iterate on $V(K)$ until convergence.

Stochastic Model (Endogenous Labor)

Now add shocks $Z \in \{0.9, 1.1\}$ with the same P_Z matrix. Expected continuation value:

$$E[V(K', Z')|Z] = \sum_{Z'} P_Z(Z'|Z) V(K', Z').$$

Repeat the nested structure:

- Inner optimization: over N for given (K, K', Z).
- Outer optimization: over K' given V and transition probabilities.

```
    for each K in K_grid
    for each Z in Z_grid
    obj_Kp = @(Kp) obj_Kp_sto(Kp, K, Z, V(:,1), V(:,2), K_grid, p, ... );
    [Kprime_opt(K,Z), Vnew(K,Z)] = goldenx(obj_Kp, Kmin, Kmax);
    [N_opt(K,Z), ~] = goldenx(@(N) u_total(C(K,N,Z,Kprime_opt(K,Z)),N,p), N_min,
        N_max);
    end
    end
```

Plots and Interpretation

Plot:

- $K'(K)$ for low and high Z.
- $N(K)$ for low and high Z.
- Value functions $V(K, Z)$.

Comment on:

- How TFP shocks affect investment and labor.
- Whether policies are monotone and smooth.

Optional Extension

Try changing ϕ or γ and describe how labor elasticity and risk aversion influence the dynamics of investment and output.